

Rafał Moderski  
*JILA, University of Colorado*  
*CB 440, Boulder, CO 80309-0440*  
*and*  
*Nicolaus Copernicus Astronomical Center*  
*Polish Academy of Sciences*  
*00-716 Warsaw, Bartycka 18, Poland*  
*moderski@camk.edu.pl*

Marek Rogatko  
*Technical University of Lublin*  
*20-618 Lublin, Nadbystrzycka 40, Poland*  
*rogat@tytan.umcs.lublin.pl*  
*rogat@akropolis.pol.lublin.pl*  
 (July 7, 1999)

04.50.+h, 98.80.Cq.

It has been argued that the extremal dilaton black holes exhibit a flux expulsion of Abelian-Higgs vortices. We re-examine carefully the problem and give analytic proofs for the flux expulsion always takes place. We also conduct numerical analysis of the problem using three initial data sets on the horizon of an extreme dilatonic black hole, namely, core, vacuum and sinusoidal initial conditions. We also show that an Abelian-Higgs vortex can end on the extremal dilaton black hole. Concluding, we calculate the backreaction of the Abelian-Higgs vortex on the geometry of the extremal black hole and drew a conclusion that a straight cosmic string and the extreme dilatonic black hole hardly knew their presence.

## I. INTRODUCTION

Wheeler's metaphoric dictum *black holes have no hair*, motivated by the earlier investigations on uniqueness theorems for black holes (see Ref. [1] and references therein), has had a great influence on the development of black hole physics. In the recent years there has been a considerable resurgence of mathematical works on black hole equilibrium states. We are faced with the discoveries of black hole solutions in many theories in which Einstein equations are coupled with self-interacting matter fields. The discovery of Bartnik and McKinnon [2] of a nontrivial point like structure in Einstein-Yang-Mills system reveals new realms of nontrivial solutions to Einstein-non-Abelian gauge systems.

Recently, the other sort of problems has been taken into account. The *no hair* conjecture has been extended to the problems when topologies of some field configurations are not trivial. The story began with the work of Aryal *et al.* [3]. Though they considered a pure gravity theory, they wrote down the metric of a black hole with a conical deficit and claimed that this was the metric describing the Schwarzschild black hole with a cosmic string passing through it. This work gave rise to the researches concerning the extension of the afore mentioned *no hair* conjecture. In Ref. [4], the extension of these considerations announced the existence of the Euclidean Einstein equations corresponding to a vortex sitting on the horizon of the black hole. Achúcarro *et al.* [5] presented the numerical and analytic evidence for an Abelian-Higgs vortex can act as a long hair for the Schwarzschild black hole. Chamblin *et al.* [6] generalized the analysis of Ref. [5] and allowed the black hole to be charged. Using numerical analysis they found that all of the fields connected with the vortex were expelled from the extremal black hole. The extremal black hole displayed an analog

of a *Meissner effect*. However, the recent works of Bonjour *et al.* [7] show subtleties in the treatment of the event horizon, showing that a flux expulsion can occur but it does not do so in all cases. Analytic proofs for an expulsion and a penetration of a flux in the case of the extremal charged black hole were also presented.

Nowadays, it seems that the superstring theories are the most promising candidates for a consistent quantum theory of gravity. Numerical studies of the solutions to the low-energy string theory, i.e., the Einstein-dilaton black holes in the presence of a Gauss-Bonnet type term, disclosed that they were endowed with a nontrivial dilaton hair [8]. This dilaton hair is expressed in terms of its ADM mass [9]. The extended moduli and dilaton hair and their associated axions for a Kerr-Newmann black hole background were computed in Ref. [10]. On the other hand, a full analysis of a cosmic string in dilaton gravity was given in Ref. [11]. An Abelian Higgs vortex in the background of an Euclidean electrically charged dilaton black hole was studied in Ref. [12]. It was shown that this kind of the Euclidean black hole can support a vortex solution on a horizon of the black hole. The vortex effect was to cut out a slice out of the considered black hole geometry. In Ref. [13] the authors argued that an electrically charged dilaton black hole could support a long range field of a Nielsen-Olesen string. Using both numerical and perturbative techniques the properties of an Abelian-Higgs vortex in the presence of the considered black hole were investigated. In the case of an extreme dilatonic black hole the analog of the *Meissner effect* was revealed.

In this paper we will try to provide some continuity with our previous works [12]- [13]. Namely, we shall re-examine the problem of the flux expulsion in the presence of the extremal dilaton black hole in the light of the arguments recently quoted by Bonjour *et al.* [7]. We have provided analytical proofs that vortices will wrap around the extreme dilatonic black hole. Then, we conduct the numerical analysis of the problem.

The paper is organized as follows. In Sec.II we briefly review some results obtained in Ref. [13] that will be needed, so that the paper becomes self-contained. We also conduct analytic considerations of the expulsion problem of the Nielsen-Olesen vortex in the presence of an extreme dilatonic black hole. In Sec.III, we give a numerical analysis of the problems taking into account three initial data sets on the horizon of the black hole, i.e., core, vacuum and sinusoidal initial conditions. We also pay attention to the problem of a vortex which terminates on the extremal dilaton black hole. Before concluding our considerations, in Sec.IV, we discuss the gravitational backreaction.

## II. NIELSEN-OLESEN VORTEX AND A DILATON BLACK HOLE

In this section we shall review some material published in Ref. [13] in order to establish notation and convention and for the paper to be self-contained. We also gave some analytic arguments in favour of a flux expulsion. In our consideration we shall study an Abelian-Higgs vortex in the presence of a dilaton black hole assuming a complete separation of degrees of freedom of each of the objects. Our system will be described by the action which is the sum of the action for a low-energy string theory [14], which in the Einstein frame, has the form

$$S_1 = \int \sqrt{-g} d^4x \left[ \frac{R}{16\pi G} - 2(\nabla\phi)^2 - e^{-2\phi} F^2 \right], \quad (1)$$

and  $S_2$  is the action for an Abelian Higgs system minimally coupled to gravity and be subject to spontaneous symmetry breaking. It yields

$$S_2 = \int \sqrt{-g} d^4x \left[ -D_\mu \Phi^\dagger D^\mu \Phi - \frac{1}{4} \tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2 \right], \quad (2)$$

where  $\Phi$  is a complex scalar field,  $D_\mu = \nabla_\mu + ieB_\mu$  is the gauge covariant derivative.  $\tilde{B}_{\mu\nu}$  is the field strength associated with  $B_\mu$ , while  $F_{\alpha\beta} = 2\nabla_{[\alpha} A_{\beta]}$ . As in Ref. [5] one can define the real fields  $X, P_\mu, \chi$  by the relations

$$\Phi(x^\alpha) = \eta X(x^\alpha) e^{i\chi(x^\alpha)}, \quad (3)$$

$$B_\mu(x^\alpha) = \frac{1}{e} [P_\mu(x^\alpha) - \nabla_\mu \chi(x^\alpha)]. \quad (4)$$

Equations of motion derived from the action  $S_2$  are given by

$$\nabla_\mu \nabla^\mu X - P_\mu P^\mu X - \frac{\lambda \eta^2}{2} (X^2 - 1)X = 0, \quad (5)$$

$$\nabla_\mu \tilde{B}^{\mu\nu} - 2e^2 \eta^2 X^2 P^\nu = 0. \quad (6)$$

The field  $\chi$  is not a dynamical quantity. In a flat spacetime the Nielsen-Olesen vortices have the cylindrically symmetric solutions of the forms

$$\Phi = X(\rho) e^{iN\phi}, \quad (7)$$

$$P_\phi = NP(\rho), \quad (8)$$

where  $\rho$  is the cylinder radial coordinate,  $N$  is the winding number.

As far as a static, spherically symmetric solution of the equations of motion derived from the action  $S_1$  is concerned it is determined by the metric of a charged dilaton black hole. The metric may be written as [15]

$$ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} + r \left( r - \frac{Q^2}{M} \right) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (9)$$

where we define  $r_+ = 2M$  and  $r_- = \frac{Q^2}{M}$  which are related to the mass  $M$  and charge  $Q$  by the relation  $Q^2 = \frac{r_+ r_-}{2} e^{2\phi_0}$ . The charge of the dilaton black hole  $Q$ , couples to the field  $F_{\alpha\beta}$ , is unrelated to the Abelian gauge field  $B_{\mu\nu}$  associated with the vortex. The dilaton field is given by  $e^{2\phi} = \left( 1 - \frac{r_-}{r} \right) e^{-2\phi_0}$ , where  $\phi_0$  is the dilaton's value at  $r \rightarrow \infty$ . The event horizon is located at  $r = r_+$ . For  $r = r_-$  is another singularity, one can however ignore it because  $r_- < r_+$ . The extremal black hole occurs when  $r_- = r_+$ , when  $Q^2 = 2M^2 e^{2\phi_0}$ .

In what follows, we shall consistently assume that  $X$  and  $P_\phi$  are functions of  $r, \theta$  coordinates. Then, equations of motion for these fields are given by

$$\frac{1}{r(r - \frac{Q^2}{M})} \partial_r \left[ \left( r - \frac{Q^2}{M} \right) (r - 2M) \partial_r X \right] + \frac{1}{r(r - \frac{Q^2}{M}) \sin \theta} \partial_\theta [\sin \theta \partial_\theta X] - \frac{N^2 P^2 X}{r(r - \frac{Q^2}{M}) \sin^2 \theta} - \frac{1}{2} X (X^2 - 1) = 0, \quad (10)$$

$$\partial_r \left[ \left( 1 - \frac{2M}{r} \right) \partial_r P \right] + \frac{\sin \theta}{r(r - \frac{Q^2}{M})} \partial_\theta [\csc \theta \partial_\theta P] - \frac{X^2 P}{\beta} = 0, \quad (11)$$

where  $\beta = \frac{\lambda}{2e^2} = m_{Higgs}^2 / m_{vect}^2$  is the Bogomolnyi parameter. When  $\beta \rightarrow \infty$ , the Higgs field decouples and like in the Reissner-Nordström case [6], one can study a free Maxwell field in the electrically charged black hole spacetime. The other situation will arise when  $P = 1$ . It will be the case of a global string in the presence of the electrically charged dilaton black hole.

One can show Ref. [16], that in normal spherically symmetric coordinates  $X$  is a function of  $\sqrt{g_{33}}$  and one will try with the coordinates  $R = \sqrt{r(r - \frac{Q^2}{M})} \sin \theta$ , namely  $X = X(R)$  and  $P_\phi = P_\phi(R)$ . Taking into account *the thin string* limit, i.e.,  $M \gg 1$ , equations for  $X(R)$  and  $P(R)$  can be reduced to the Nielsen-Olesen type up to the errors of an adequate order (see Ref. [13]).

The main task of our work will be to answer the question about the flux expulsion in the case of the extremal dilaton black hole. Some numerical arguments concerning the so-called *Meissner effect* were quoted in Ref. [6]. Now, having in mind the arguments of Ref. [7], we examine this problem carefully. We begin with the analytical considerations. From now on, we shall consider only the extremal dilaton black hole, for which  $r_+ = r_-$ .

As one can see from Eqs.(10-11), the flux expulsion solution  $X = 0$ ,  $P = 1$  always solves these equations of motion. Thus, our strategy will be to show the nonexistence of a penetration solution. First of all, we assume that a piercing solution does exist. This requirement is fulfilled when one has the nontrivial solutions  $X(\theta)$  and  $P(\theta)$  which is symmetric around  $\theta = \frac{\pi}{2}$ . For this value of the angle  $X$  has a maximum and  $P$  a minimum. Expanding equations of motion for  $P(\theta)$  near the poles indicates that  $P_{,\theta} = 0$  at the poles. Thus, there exists such a point  $\theta_0 \in (0, \frac{\pi}{2})$  for which the second derivative of  $P(\theta)$  is equal to zero, namely  $P_{,\theta\theta}(\theta_0) = 0$  and  $P_{,\theta}(\theta_0) < 0$ . In the extremal black hole case Eq.(11) has the form as follows

$$P_{,\theta\theta} - P_{,\theta} \cot \theta = 0. \quad (12)$$

Having in mind that  $P_{,\theta\theta}(\theta_0) = 0$  and  $\cot \theta_0 \neq 0$  in the considered interval of  $\theta$ , we reach to the contradiction with our starting assumption that  $P_{,\theta}(\theta_0) < 0$ . This is sufficient to conclude that the flux expulsion must always take place.

Further, after proving that the flux expulsion must take place for a sufficiently thick string, we proceed to the case of a thin string. As was remarked in Ref. [7] in order to consider the case of a thin string one has to look at the Eqs.(10-11) in the exterior region of the horizon. To begin with, we assume that there is a flux expulsion. Therefore near the horizon of the extremal dilaton black hole, one has  $\partial_r[(r - 2M)^2 \partial_r X] > 0$ ,  $X^3 2M(2M - \frac{Q^2}{M}) \ll 1$ . Hence from Eq. (10) we obtain

$$\sin^2 \theta \partial_\theta [(r - 2M)^2 \partial_r X] + \sin \theta \partial_\theta [\sin \theta \partial_\theta X] - X N^2 P^2 = 0. \quad (13)$$

The function  $X$  is symmetric around  $\frac{\pi}{2}$ , having maximum  $X_m$ . Integrating Eq.(13) on the interval  $(\theta, \frac{\pi}{2})$ , for  $\theta > \beta$  we arrive at the inequality

$$\partial_\theta X(\theta) > X(\theta) \left[ \frac{N^2}{\sin \theta} \ln \tan \frac{\theta}{2} \right]. \quad (14)$$

Then using the fact that  $X_{,\theta\theta} < 0$  on  $[\theta_0, \frac{\pi}{2}]$ , one can deduce that  $X_{,\theta} < \frac{X(\theta) - X(\theta_0)}{\theta - \theta_0} < \frac{X(\theta)}{\theta - \beta}$ . This shows that the inequality

$$\frac{1}{N^2} > (\theta - \beta) \frac{1}{\sin \theta} \ln \tan \frac{\theta}{2}, \quad (15)$$

must hold over the range  $\theta \in (\beta, \frac{\pi}{2})$  for the expulsion to occur. From the graph of the function  $\zeta(\theta) = \frac{1}{\sin \theta} \ln \tan \frac{\theta}{2}$ , we deduce that on the interval  $\theta \in (\beta, \frac{\pi}{2})$  one has  $\zeta(\theta) < 0$ . Then, the inequality (15) always holds and we have the expulsion of the vortex for the extremal dilaton black hole.

The analytic solution of equations of motion for the dilaton black hole which size was small comparing to the vortex size were considered by the present authors in Ref. [13]. It was done for the large  $N$ -limit. Then it happened that, inside a core of the vortex the gauge symmetry will be unbroken, and the expectation that  $\frac{X^2}{\beta} \approx 0$  is well justified. These all provided the solution of Eq.(11) as

$$P \approx 1 - pR \sin^2 \theta, \quad (16)$$

where  $p$  is an integration constant equal to twice the magnetic field strength at the center of the core [5]. On the other hand, the solution for  $X$  yields

$$X \approx b^N(r) \sin^N \theta. \quad (17)$$

where  $b(r)$  is given by Eq.(21) in Ref. [13]. The exact form of  $X$  ensures its vanishing on the extremal black hole horizon. Concluding we see that, using analytical arguments one has always the expelling of the fields from the extreme dilaton black hole horizon. As we see in the next section these analytical considerations are fully confirmed by numerical investigations.

### III. NUMERICAL RESULTS

To confirm our results from previous section we performed number of numerical calculations for extremal dilaton black holes with strings. The numerical method is simply the same as in our previous article [13] (see also Ref. [17]). Namely, the fields  $X$  and  $P$  are replaced with their discretized values on the polar grid  $(r, \theta) \rightarrow (r_i = 2M + i\Delta r, \theta_k = k\Delta\theta)$  according to the difference version of the equations of motion (10) and (11)

$$X_{00} = \frac{(1 - \frac{2M}{r}) \frac{X_{+0} + X_{-0}}{(\Delta r)^2} + \frac{1}{r(r - \frac{Q^2}{M})} \frac{X_{0+} + X_{0-}}{(\Delta\theta)^2} + \left[ \frac{1}{r} + \frac{r - 2M}{r(r - \frac{Q^2}{M})} \right] \frac{X_{+0} - X_{-0}}{2\Delta r} + \frac{\cot \theta}{r(r - \frac{Q^2}{M})} \frac{X_{0+} - X_{0-}}{2\Delta\theta}}{(1 - \frac{2M}{r}) \frac{2}{(\Delta r)^2} + \frac{2}{r(r - \frac{Q^2}{M})(\Delta\theta)^2} + \frac{1}{r(r - \frac{Q^2}{M})} \left( \frac{P_{00}N}{\sin \theta} \right)^2 + \frac{1}{2}(X_{00}^2 - 1)} \quad (18)$$

$$P_{00} = \frac{(1 - \frac{2M}{r}) \frac{P_{+0} + P_{-0}}{(\Delta r)^2} + \frac{1}{r(r - \frac{Q^2}{M})} \frac{P_{0+} + P_{0-}}{(\Delta\theta)^2} + \frac{2M}{r} \frac{P_{+0} - P_{-0}}{2\Delta r} - \frac{\cot \theta}{r(r - \frac{Q^2}{M})} \frac{P_{0+} - P_{0-}}{2\Delta\theta}}{(1 - \frac{2M}{r}) \frac{2}{(\Delta r)^2} + \frac{2}{r(r - \frac{Q^2}{M})(\Delta\theta)^2} + \frac{1}{\beta} X_{00}^2}, \quad (19)$$

On the horizon we used

$$X_{00} = \frac{\frac{1}{\Delta r} X_{+0} + \frac{\cot \theta}{(2M - \frac{Q^2}{M})} \frac{X_{0+} - X_{0-}}{2\Delta\theta} + \frac{1}{(2M - \frac{Q^2}{M})} \frac{X_{0+} + X_{0-}}{(\Delta\theta)^2}}{\frac{1}{\Delta r} + M(X_{00}^2 - 1) + \frac{1}{(2M - \frac{Q^2}{M})} \left( \frac{P_{00}N}{\sin \theta} \right)^2 + \frac{2}{(2M - \frac{Q^2}{M})(\Delta\theta)^2}} \quad (20)$$

$$P_{00} = \frac{\frac{1}{\Delta r} P_{+0} - \frac{\cot \theta}{(2M - \frac{Q^2}{M})} \frac{P_{0+} - P_{0-}}{2\Delta\theta} + \frac{1}{(2M - \frac{Q^2}{M})} \frac{P_{0+} + P_{0-}}{(\Delta\theta)^2}}{\frac{1}{\Delta r} + \frac{1}{\beta} 2M X_{00}^2 + \frac{2}{(2M - \frac{Q^2}{M})(\Delta\theta)^2}} \quad (21)$$

The boundary conditions are imposed on large radii and for a string core according to

$$(X, P) = \begin{cases} (1, 0), & r \rightarrow \infty \\ (0, 1), & r \geq 2M, \theta = 0, \pi. \end{cases} \quad (22)$$

Boundary conditions on the horizon are guessed at the begining of calculation and then updated in accordance with Eqs. (20) and (21). The process of relaxation and updating of the fields on the horizon of the black hole is repeated until convergence take place.

As an initial guess we used three sets of different boundary conditions as proposed by Bonjour *et al.* [7], i.e.,

1. core:  $X = 0, P = 1$ ,
2. vacuum:  $X = 1, P = 0$ ,
3. sinusoidal:  $X(\theta) = \sin \theta, P = 1$ .

In all above cases we received the same final configuration of the fields always showing the string expulsion from the extreme dilatonic black hole. These confirm our previous theoretical predictions. The results of the numerical calculations are presented in Fig. 1.

Further, we pay our attention to the case of a cosmic string ending on the extreme dilatonic black hole. This is an important configuration as the main phenomenological input to the instantons mediating defect decays (see, e.g., [18]-[20]). Namely, in Ref. [16] it was shown that the Nielsen-Olesen solution could be used to construct regular metrics which represented vortices ended on black holes either in a static equilibrium or accelerating off to infinity. The latter metric depicts a cosmic string which is eaten by accelerating black holes.

To consider numerically a string ending on the black hole we have to slightly modify the boundary conditions (22). They remain the same at the outer boundary and in the string core for  $\theta = 0$ . For  $\theta = \pi$  we initially set  $X = 1$  and  $P = 0$ . During the calculation values of the fields were updated on each step according to  $P_{,\theta} = 0$  and  $X_{,\theta} = 0$ . These were chosen to assure the numerical stability. On the horizon we set vacuum boundary conditions to avoid the possibility that flux expulsion, if it occurs, may be caused by the inappropriate choice of initial conditions on the horizon. The results of our simulations for extreme dilatonic black hole with  $M = 1.0$  and  $\beta = N = 1$  are presented in Fig. 2.

#### IV. GRAVITATIONAL BACKREACTION

The gravitational backreaction of the string on the dilaton black hole geometry was studied in Ref. [13]. As one can compare the resultant metric we shall obtain and the conical metric gained in Ref. [13] they are the same. But in our previous attitude we did not pay attention to the corrections of other fields in the theory. Therefore our previous results were not correct. In this section we consider the backreaction problem taking into account all the fields in the theory.

In order to consider the gravitational backreaction effect of the string superimposed on the dilaton black hole one needs to consider a general static axially symmetric solution to the Einstein-Maxwell-dilaton Abelian-Higgs equations of motion. First, we find a coordinate transformation which enables us to write the *spherical* dilaton black hole metric in the axisymmetric form. Using the coordinate transformation as follows:

$$\rho^2 = (r - 2M) \left( r - \frac{Q^2}{M} \right) \sin^2 \theta, \quad (23)$$

$$z = \left( r - M - \frac{Q^2}{2M} \right) \cos \theta, \quad (24)$$

the metric of the charged dilaton black hole may be rewritten in the axisymmetric form, namely

$$ds^2 = -e^{2\psi_0} dt^2 + \alpha_0^2 e^{-2\psi_0} d\varphi^2 + e^{-2\psi_0 + 2\gamma_0} (d\rho^2 + dz^2), \quad (25)$$

where

$$e^{-2\psi_0} = \frac{r}{r-2M}, \quad (26)$$

$$e^{-2\psi_0+2\gamma_0} = \frac{\left(r - \frac{Q^2}{M}\right)r}{A}, \quad (27)$$

$$\rho^2 = \alpha_0^2, \quad (28)$$

$$A = (r-2M) \left(r - \frac{Q^2}{M}\right) \cos^2 \theta + \frac{\sin^2 \theta}{4} \left[ (r-2M) + \left(r - \frac{Q^2}{M}\right) \right]^2. \quad (29)$$

The dilaton black hole metric (25) will constitute, to the zeroth order, our background solution. The relevant equations of motion become

$$\partial_\rho (e^{-2\phi-2\psi} \alpha F_{0\rho}) + \partial_z (e^{-2\phi-2\psi} \alpha F_{0z}) = 0, \quad (30)$$

$$\partial_\rho (\alpha \partial_\rho \phi) + \partial_z (\alpha \partial_z \phi) + e^{-2\phi} \sqrt{-g} [(F_{0\rho})^2 + (F_{0z})^2] = 0, \quad (31)$$

$$\alpha_{,zz} + \alpha_{,\rho\rho} = \sqrt{-g} (\mathcal{T}_z^z + \mathcal{T}_\rho^\rho), \quad (32)$$

$$(\alpha \psi_{,z})_{,z} + (\alpha \psi_{,\rho})_{,\rho} = -\frac{1}{2} \sqrt{-g} (\mathcal{T}_0^0 - \mathcal{T}_z^z - \mathcal{T}_\rho^\rho - \mathcal{T}_\varphi^\varphi), \quad (33)$$

$$-\gamma_{,\rho} (\alpha_{,\rho}^2 + \alpha_{,z}^2) + \alpha \alpha_{,\rho} (\psi_{,\rho}^2 - \psi_{,z}^2) + 2\alpha \alpha_{,z} \psi_{,\rho} \psi_{,z} + \alpha_{,\rho} \alpha_{,\rho\rho} + \alpha_{,z} \alpha_{,\rho z} = \sqrt{-g} (\alpha_{,\rho} \mathcal{T}_z^z - \alpha_{,z} \mathcal{T}_\rho^\rho), \quad (34)$$

$$\gamma_{,z} (\alpha_{,\rho}^2 + \alpha_{,z}^2) - \alpha \alpha_{,z} (\psi_{,\rho}^2 - \psi_{,z}^2) - 2\alpha \alpha_{,\rho} \psi_{,\rho} \psi_{,z} + \alpha_{,z} \alpha_{,\rho\rho} - \alpha_{,\rho} \alpha_{,\rho z} = \sqrt{-g} (\alpha_{,z} \mathcal{T}_z^z - \alpha_{,\rho} \mathcal{T}_\rho^\rho), \quad (35)$$

$$\gamma_{,zz} + \gamma_{,\rho\rho} + \psi_{,z}^2 + \psi_{,\rho}^2 = \sqrt{-g} (e^{2\gamma-2\psi} \mathcal{T}_\varphi^\varphi), \quad (36)$$

where the energy momentum tensor  $\mathcal{T}_\alpha^\beta$  is given by

$$\mathcal{T}_\alpha^\beta = \epsilon T_\alpha^\beta(string) + T_\alpha^\beta(F, \phi), \quad (37)$$

where  $\epsilon = 8\pi G \eta^2$  which is assumed to be small. This assumption is well justified because, e.g., for the GUT string  $\epsilon \leq 10^{-6}$ . The first term is the contribution from the string and it has the following explicit form:

$$T_0^0(string) = -e^{-2(\gamma-\psi)} (X_{,\rho}^2 + X_{,z}^2) - \frac{X^2 P^2 e^{2\psi}}{\alpha^2} - \frac{\beta}{\alpha^2} e^{-2\gamma+4\psi} (P_{,\rho}^2 + P_{,z}^2) - V(X), \quad (38)$$

$$T_\varphi^\varphi(string) = -e^{-2(\gamma-\psi)} (X_{,\rho}^2 + X_{,z}^2) + \frac{X^2 P^2 e^{2\psi}}{\alpha^2} + \frac{\beta}{\alpha^2} e^{-2\gamma+4\psi} (P_{,\rho}^2 + P_{,z}^2) - V(X), \quad (39)$$

$$T_\rho^\rho(string) = e^{-2(\gamma-\psi)} (X_{,\rho}^2 - X_{,z}^2) - \frac{X^2 P^2 e^{2\psi}}{\alpha^2} + \frac{\beta}{\alpha^2} e^{-2\gamma+4\psi} (P_{,\rho}^2 - P_{,z}^2) - V(X), \quad (40)$$

$$T_z^z(string) = e^{-2(\gamma-\psi)} (X_{,z}^2 - X_{,\rho}^2) - \frac{X^2 P^2 e^{2\psi}}{\alpha^2} + \frac{\beta}{\alpha^2} e^{-2\gamma+4\psi} (P_{,z}^2 - P_{,\rho}^2) - V(X), \quad (41)$$

$$T_\rho^z(string) = 2e^{-2(\gamma-\psi)} \left( X_{,\rho} X_{,z} + \frac{\beta}{\alpha^2} P_{,\rho} P_{,z} \right), \quad (42)$$

where  $V(X) = \frac{(X^2-1)^2}{4}$ . The electromagnetic dilaton contribution is given by

$$T_{\mu\nu}(F, \phi) = e^{-2\phi} (4F_{\mu\rho}F_{\nu}{}^{\rho} - g_{\mu\nu}F^2) - 2g_{\mu\nu}(\nabla\phi)^2 + \nabla_{\mu}\phi\nabla_{\nu}\phi. \quad (43)$$

Having in mind Eq.(43) one notices that the electromagnetic-dilaton energy momentum tensor always fulfills

$$T_{\rho}{}^{\rho}(F, \phi) + T_z{}^z(F, \phi) = 0. \quad (44)$$

As in Ref. [5] we will solve the Einstein-Maxwell-dilaton equations iteratively, i.e.,  $\alpha = \alpha_0 + \epsilon\alpha_1$  etc. Following Ref. [13] we can use the coordinates  $R = \sqrt{r(r - \frac{Q^2}{M})} \sin\theta = \rho e^{-\psi_0}$ , which yields that near the core of the string where  $\sin\theta \approx \mathcal{O}(M^{-1})$ , one gets  $R_{,\rho}^2 + R_{,z}^2 \sim e^{2\gamma_0 - 2\psi_0}$ . This relation implies that near the core of the string, to the zeroth order, the energy momentum tensor reads as follows:

$$T_{(0)0}0(string) = -V(X_0) - \left(\frac{d}{dR}X_0\right)^2 - \frac{X_0^2 P_0^2}{R^2} - \frac{\beta}{R^2} \left(\frac{d}{dR}P_0\right)^2, \quad (45)$$

$$T_{(0)\varphi}\varphi(string) = -V(X_0) - \left(\frac{d}{dR}X_0\right)^2 + \frac{X_0^2 P_0^2}{R^2} + \frac{\beta}{R^2} \left(\frac{d}{dR}P_0\right)^2, \quad (46)$$

$$T_{(0)\rho}\rho(string) = -V(X_0) - \left(\frac{d}{dR}P_0\right)^2 + e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left(\frac{d}{dR}P_0\right)^2 + \left(\frac{d}{dR}X_0\right)^2 \right] (R_{,\rho}^2 - R_{,z}^2), \quad (47)$$

$$T_{(0)z}z(string) = -V(X_0) - \left(\frac{d}{dR}P_0\right)^2 - e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left(\frac{d}{dR}P_0\right)^2 + \left(\frac{d}{dR}X_0\right)^2 \right] (R_{,\rho}^2 - R_{,z}^2), \quad (48)$$

$$T_{(0)\rho}z(string) = 2e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left(\frac{d}{dR}P_0\right)^2 + \left(\frac{d}{dR}X_0\right)^2 \right] R_{,\rho}R_{,z}. \quad (49)$$

which is the purely function of  $R$ . As in the Schwarzschild case [5], this strongly suggests to look for the metric perturbations as a function of  $R$ .

As in Ref. [7] we assume that the first order perturbed solutions take form

$$\alpha_1 = \rho a(R), \quad \psi_1 = \psi_1(R), \quad \gamma_1 = \gamma_1(R), \quad \phi_1 = \phi_1(R), \quad A_{\mu}^{(1)} = f(R)A_{\mu}^{(0)}. \quad (50)$$

Further, carry out the computing of the necessary derivatives one gets the following equation for  $a(R)$

$$\frac{d^2}{dR^2}a(R) + \frac{2}{R}\frac{d}{dR}a(R) = T_{\rho}{}^{\rho}(string) + T_z{}^z(string) + \mathcal{O}(M^{-2}). \quad (51)$$

From which one can reach to the following expression for  $a(R)$ :

$$a(R) = \int_R^{\infty} \frac{1}{R'^2} dR' \int_0^{R'} R'^2 \left( -2V(X_0) - \frac{2X_0^2 P_0^2}{R'^2} \right) dR'. \quad (52)$$

Eq. (52) can be rewritten as

$$a(R) \sim -A + \frac{B}{R}, \quad (53)$$

where

$$A = \int_0^R R(T_{(0)\rho}{}^{\rho}(string) + T_{(0)z}{}^z(string))dR, \quad (54)$$

$$B = \int_0^R R(T_{(0)\rho}{}^{\rho}(string) + T_{(0)z}{}^z(string))dR. \quad (55)$$



On the other hand, Eq.(30) implies

$$\frac{d^2}{dR^2}f + \frac{1}{R^2}\frac{d}{dR}f = \frac{\rho^2}{R^2r^2} \left[ \frac{d}{dR}a - 2 \left( \frac{d}{dR}\phi_1 + \frac{d}{dR}\psi_1 \right) \right] = \mathcal{O}(M^{-2}), \quad (56)$$

which yields that  $f = f_0$  is equal to a constant value. The magnetic correction one can get either directly or using the duality transformation [22], which implies  $F \rightarrow *F$ ,  $\phi \rightarrow -\phi$ , where  $*F_{\mu\nu} = \frac{1}{2}e^{-2\phi}\epsilon_{\mu\nu\rho\delta}F^{\rho\delta}$ . Turning our attention to Eq.(33) and taking into account the value of  $f_0$ , one finds for  $\psi_1$

$$\frac{d^2}{dR^2}\psi_1 + \frac{1}{R}\frac{d}{dR}\psi_1 = -\frac{1}{2} (T_{(0)0}{}^0 - T_{(0)z}{}^z - T_{(0)\rho}{}^\rho - T_{(0)\varphi}{}^\varphi) (string) + \frac{4\rho^2}{R^2} \left( \frac{Q^2}{r^4} \right) [f_0 - (\psi_1 + \phi_1)]. \quad (57)$$

The result of the integration of Eq.(57) is given by

$$\psi_1(R) = -\frac{1}{2} \int_R^\infty \frac{1}{R} dR \int_0^R R' \left( 2V(X_0) - \frac{2\beta}{R'^2} \left( \frac{dP_0}{dR'} \right)^2 \right) dR'. \quad (58)$$

Consider now Eq.(36). For  $\gamma_1(R)$ , one gets the expression of the form

$$\gamma_1(R) = \int_R^\infty dR \int_0^R T_{\varphi}{}^\varphi(string) dR' = 2\psi_1(R). \quad (59)$$

Similarly, for  $\phi_1$  we arrive at the expression

$$\frac{d^2}{dR^2}\phi_1 + \frac{1}{R}\frac{d}{dR}\phi_1 = \frac{\rho^2}{R^2} \left( \frac{Q^2}{r^4} \right) [f_0 - (\psi_1 + \phi_1 + \gamma_1)], \quad (60)$$

which implies that  $\phi_1 = \tilde{\phi} \ln \sqrt{r(r - \frac{Q^2}{M})} \sin \theta$ , where  $\tilde{\phi}$  is a constant value.

Taking into account the above corrections, one can consistently transform the metric to the  $(t, r, \theta, \varphi)$  coordinates in which the asymptotic form of the metric is expressed as

$$ds^2 \rightarrow e^{\epsilon C} \left[ - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} + r \left( r - \frac{Q^2}{M} \right) d\theta^2 \right] + \\ + r \left( r - \frac{Q^2}{M} \right) \left[ 1 - \epsilon A + \frac{\epsilon B}{\sqrt{r(r - \frac{Q^2}{M})} \sin \theta} \right]^2 e^{-\epsilon C} \sin^2 \theta d\varphi^2, \quad (61)$$

where  $e^{\epsilon C} = e^{2\epsilon\psi_1}$ .

One should notice [5] that the  $B$ -term is outside the range of the applicability of the considered approximation. After rescaling coordinates  $\hat{t} = e^{\epsilon C/2}t$ ,  $\hat{r} = e^{\epsilon C/2}r$  and defining the quantities  $\hat{M} = e^{\epsilon C/2}M$  and  $\hat{Q} = e^{\epsilon C/2}Q$ , one gets the metric in the *thin string* metric, i.e.  $M \gg 1$ ,

$$ds^2 = - \left( 1 - \frac{2\hat{M}}{\hat{r}} \right) d\hat{t}^2 + \frac{d\hat{r}^2}{\left( 1 - \frac{2\hat{M}}{\hat{r}} \right)} + \hat{r} \left( \hat{r} - \frac{\hat{Q}^2}{\hat{M}} \right) d\theta^2 + \hat{r} \left( \hat{r} - \frac{\hat{Q}^2}{\hat{M}} \right) (1 - \epsilon A)^2 e^{-2\epsilon C} \sin^2 \theta d\varphi^2. \quad (62)$$

Now, we turn to a deficit angle, which has the form  $\delta\theta = 2\pi\epsilon(A + C) = 8\pi G\mu$ , where  $\mu$  is the total mass of string per unit length. On the other hand, its ADM mass generalized to an asymptotically flat space is written [21]

$$M_I = \hat{M}(1 - \epsilon A)e^{-\epsilon C}. \quad (63)$$

The definition of the physical charge of the black hole, respectively for magnetic or electric charge is given by

$$Q_{ph} = \frac{1}{4\pi} \int_{S^2} F_{\mu\nu} dS^{\mu\nu}, \quad Q_{ph} = \frac{1}{4\pi} \int_{S^2} *F_{\mu\nu} dS^{\mu\nu}, \quad (64)$$

where  $dS_{\mu\nu} = l_{[\mu} n_{\nu]} dA$  and  $dA$  is the element of surface area. The null vector  $n_\mu$  is orthogonal to the two-sphere on the horizon, with the normalization condition  $l_\alpha n^\alpha = -1$ .

Then, one can write the first order corrected solutions in terms of  $M_I$  and  $Q_{ph}$ . The resultant metric has the form

$$ds^2 = - \left( 1 - \frac{2M_I e^{4G\mu}}{\hat{r}} \right) dt^2 + \frac{d\hat{r}^2}{\left( 1 - \frac{2M_I e^{4G\mu}}{\hat{r}} \right)} + \hat{r} \left( \hat{r} - \frac{Q_{ph}^2 e^{4G\mu - 2\epsilon\tilde{D}}}{M_I} \right) d\theta^2 \\ + \hat{r} \left( \hat{r} - \frac{Q_{ph}^2 e^{4G\mu - 2\epsilon\tilde{D}}}{M_I} \right) e^{-4G\mu} \sin^2 \theta d\varphi^2, \quad (65)$$

where  $\tilde{D} = 2\phi_1 + f_0$ . The corrected inner  $\hat{r}_-$  and outer  $\hat{r}_+$  horizons are situated at

$$\hat{r}_- = \frac{Q_{ph}^2 e^{4G\mu - 2\epsilon\tilde{D}}}{M_I}, \quad \hat{r}_+ = 2M_I e^{4G\mu}. \quad (66)$$

On the other hand, the corrected condition for the extremal black hole yields

$$2M_I^2 = Q_{ph}^2 e^{-2\epsilon\tilde{D}}. \quad (67)$$

Finally, using the formula for the entropy [21], we conclude that it has form as follows:

$$S = 2\pi M_I e^{6G\mu} \left( 2M_I - \frac{Q_{ph} e^{-2\epsilon\tilde{D}}}{M_I} \right). \quad (68)$$

Finally, we will discuss another interesting problem that can be explored, i.e., the interaction of a cosmic string with an extreme dilatonic black hole. We shall consider the vortex which is in a perfect alignment with an extremal black hole axis. This assumptions enables one to get rid of great complications when the black hole and the vortex are displaced to each other. We also assume that, we do not take into account details of the core structure of a cosmic string. It is turned out that the extreme dilatonic black hole metric can be written as [23]

$$ds^2 = -\frac{1}{V} dt^2 + V(dx^2 + dy^2 + dz^2) \\ A_\mu = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{V} \right) \partial_\mu t, \quad (69)$$

where the function  $V$  satisfies  $\nabla^2_{(x,y,z)} V = 0$ . In the cylindrical coordinates  $(\rho, z, \varphi)$  centered on the string, with a conical deficit  $0 \leq \varphi \leq \frac{2\pi}{p}$ ,  $p \approx 1 + 4\mu G$ , the function  $V$  takes the form (see, e.g., Ref. [24])

$$V(z, \rho, \varphi; \rho_0) = 1 + \frac{2M}{\pi\sqrt{2\rho\rho_0}} \int_{u_0}^{\infty} \frac{du}{\sqrt{\cosh u - \cosh u_0}} \frac{p \sinh pu}{\cosh pu - \cos p\varphi}, \quad (70)$$

where we set the black hole at  $\rho = \rho_0, \varphi = 0$ , and  $z = 0$ . While  $u_0$  is defined by the relation  $\cosh u_0 = \frac{\rho^2 + z^2 + \rho_0^2}{2\rho\rho_0}$ . The obtained function is nonsingular away from the conical line and the singularity of the black hole. Then, there are no forces between the extremal black hole and the cosmic string. Analogous results have been obtained in the case of a cosmic string and an extremal Reissner-Nordström black hole [7]. Of course, we should be aware of neglecting the effect of the dilaton extremal black hole on the string core. Nevertheless, using our assumption of a complete separation of degrees of freedom of each of the objects one concludes that an extremal dilaton black hole and a straight cosmic string will hardly feel their presence.

In our work we ask the question of whether or not an Abelian-Higgs vortex is expelled from the extremal dilaton black hole. We gave analytical arguments that no matter how thick was the vortex it was always expelled from the considered black hole. In order to confirm our analytic results we performed numerical calculations in which boundary conditions on the horizon of the extreme dilatonic black hole were guessed at the beginning of the process and updated according to the adequate equations. We also paid attention to the vortices ending on the extremal dilaton black hole.

Finally, we studied the backreaction effect of the vortex on the geometry and the other fields in the theory under consideration. In *the thin string* limit we get the *conical* dilaton black hole metric. Concluding, we mentioned the problem of an interaction between a straight cosmic string and an extremal dilaton black hole which was situated in the perfect alignment with a black hole axis. According to the assumptions of the clear separation of the degrees of freedom of these objects, one can conclude that they hardly feel their presence.

### Acknowledgements:

We would like to thank Ruth Gregory and Roberto Emparan for helpful remarks and discussions on various occasions. R.M. acknowledges support from the NASA Long Term Astrophysics grant NASA-NAG-6337 and NSF grant No. AST-9529175.

- 
- [1] M.Heusler, *Black Holes Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996),  
K.S.Thorne, *Black Holes and Time Warps* (W.W.Norton and Company, New York, 1994).
  - [2] R.Bartnik and J.McKinnon, Phys. Rev. Lett. **61**, 41 (1988).
  - [3] M.Aryal, L.H.Ford, and A.Vilenkin, Phys. Rev. D **34**, 2263 (1986).
  - [4] F.Dowker, R.Gregory, and J.Traschen, Phys. Rev. D **45**, 2762 (1992).
  - [5] A.Achúcarro, R.Gregory, and K.Kuijken, Phys. Rev. D **52**, 5729 (1995).
  - [6] A.Chamblin, J.M.A.Ashbourn-Chamblin, R.Empanan, and A.Sornborger, Phys. Rev. D **58**, 124014 (1998),  
A.Chamblin, J.M.A.Ashbourn-Chamblin, R.Empanan, and A.Sornborger, Phys. Rev. Lett. **80**, 4378 (1998).
  - [7] F.Bonjour and R.Gregory, Phys. Rev. Lett. **81**, 5034 (1998),  
F.Bonjour, R.Empanan and R.Gregory, Phys. Rev. D **59**, 084022 (1999).
  - [8] P.Kanti, N.E.Mavromatos, J.Rizos, K.Tamvakis, and E.Winstanley, Phys. Rev. D **54**, 5049 (1996).
  - [9] R.Arnowitt, S.Deser, and C.Misner in *Gravitation: an Introduction to Current Research*, ed.L.Witten (Wiley, New York 1962).
  - [10] P.Kanti and K.Tamvakis, Phys. Rev. D **52**, 3506 (1995).
  - [11] R.W.Gregory and C.Santos Phys. Rev. D **56**, 1194 (1998).
  - [12] R.Moderski and M.Rogatko, Phys. Rev. D **57**, 3449 (1998).
  - [13] R.Moderski and M.Rogatko, Phys. Rev. D **58**, 124016 (1998).
  - [14] D.J.Gross and J.H.Sloan, Nucl. Phys. B **291**, 41 (1987),  
C.G.Callan, R.C.Myers, and M.J.Perry, Nucl. Phys. B **311**, 673 (1988).

- [15] D.Garfinkle, G.T.Horowitz, and A.Strominger, Phys. Rev. D **43**, 3140 (1991),  
Phys. Rev. D **45**, 3888 (1992) (Erratum).
- [16] R.Gregory and M.Hindmarsh, Phys. Rev. D **52**, 5598 (1995).
- [17] W.H.Press, B.R.Flannery, S.A.Teukolsky, and W.T.Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1988).
- [18] D.Eardeley, G.Horowitz, D.Kastor, and J.Trashen, Phys. Rev. Lett. **75**, 3390 (1995).
- [19] S.W.Hawking and S.F.Ross, Phys. Rev. Lett. **75**, 3382 (1995).
- [20] R.Empanan, Phys. Rev. Lett. **75**, 3386 (1995),  
Phys. Rev. D **52**, 6976 (1995).
- [21] S.W.Hawking and G.T.Horowitz, Class. Quantum Grav. **13**, 1487 (1996).
- [22] G.Horowitz, "The Dark Side of String Theory - Black Holes and Black Strings", hep-th **9210119** (1992).
- [23] K.Siraishi, Class. Quantum Grav. **13**, 1655 (1996).
- [24] A.G.Smith, in *The Formation and Evolution of Cosmic Strings*, ed.G.W.Gibbons, S.W.Hawking, and T.Vachaspati (Cambridge University Press, Cambridge, England, 1990).

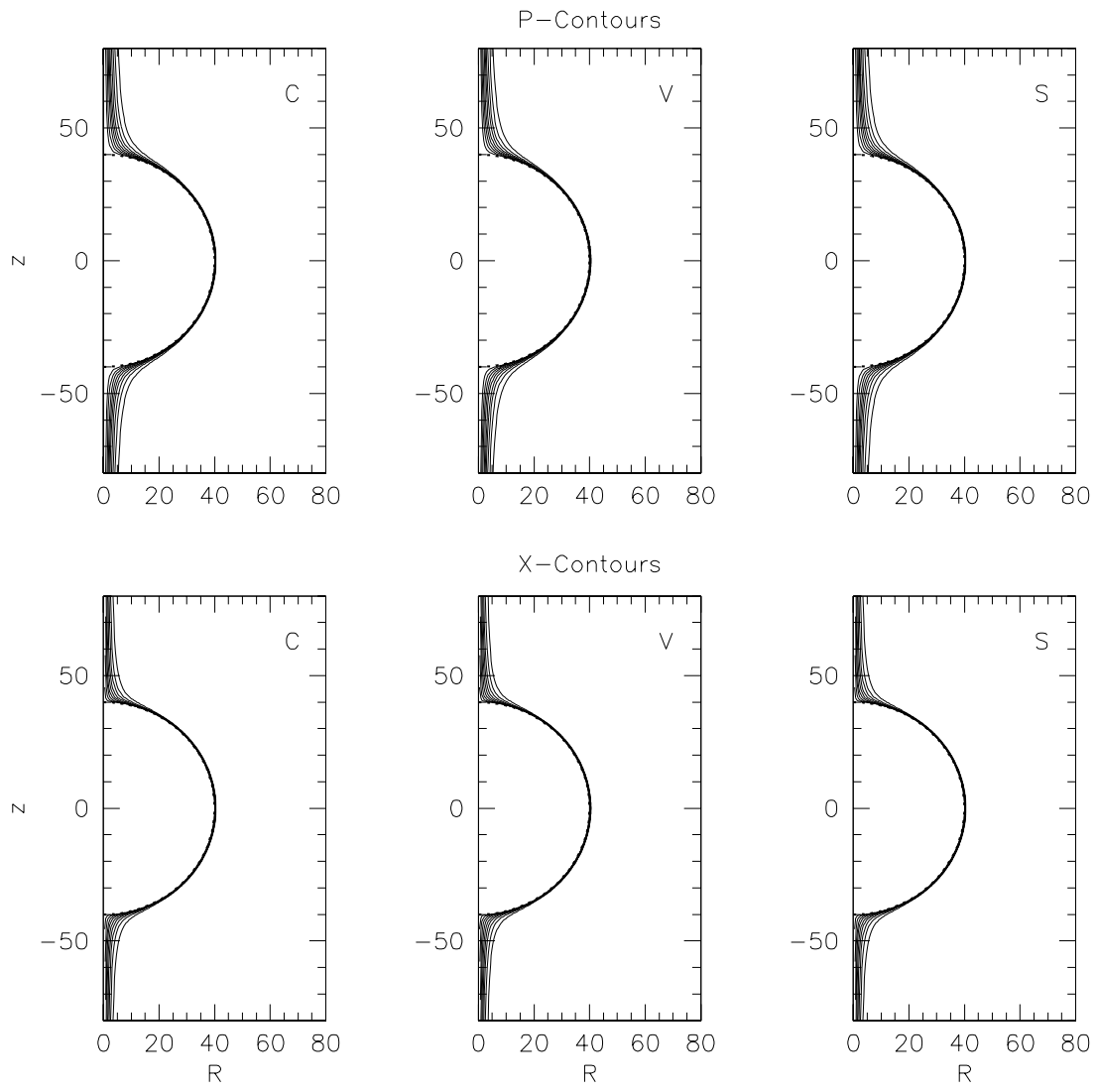


FIG. 1. Contours of  $X$  and  $P$  for a core (C), vacuum (V), and sinusoidal (S) initial guess. The parameters  $M = 1, Q = 1.41M, \beta = N = 1$ .

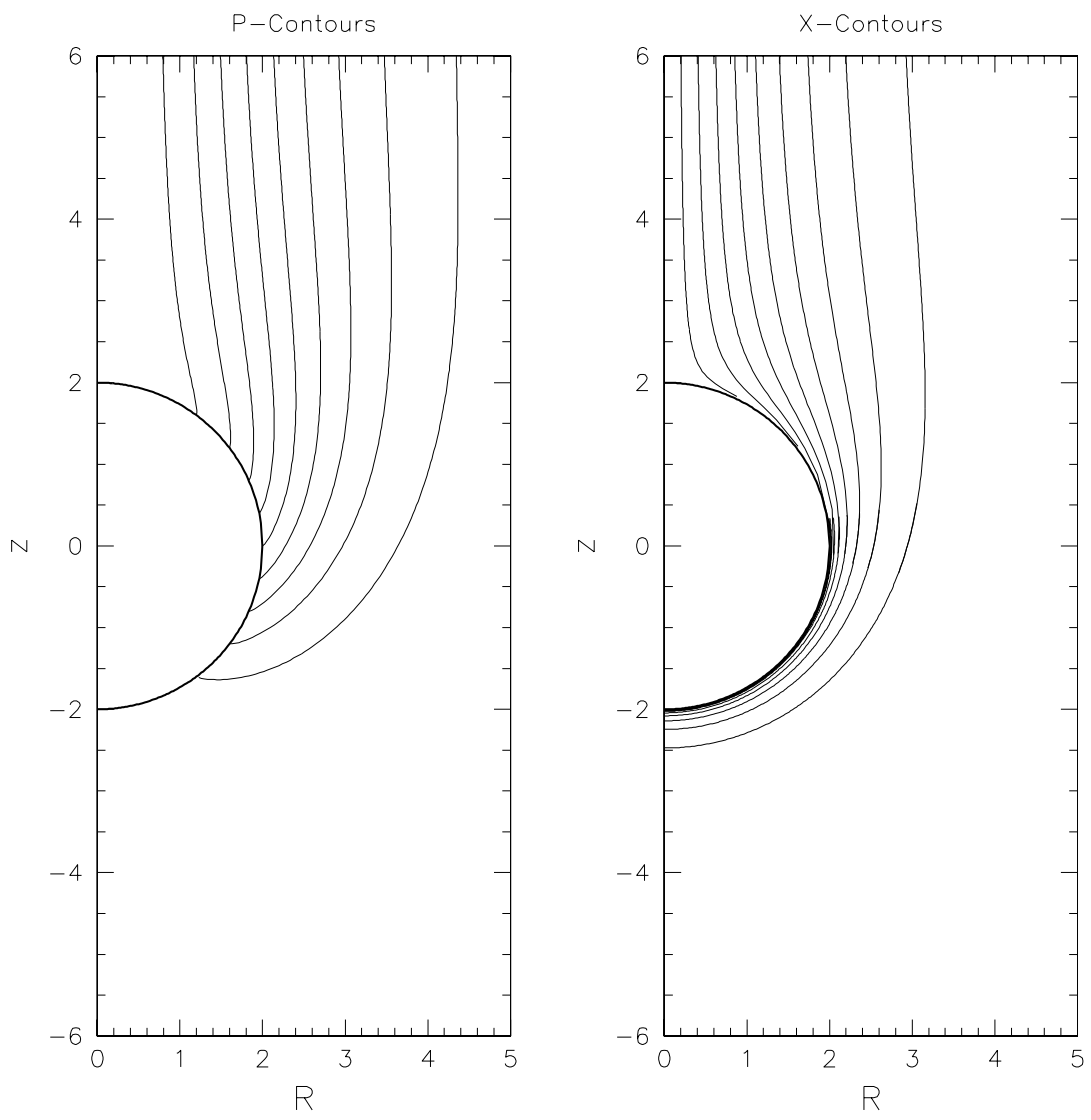


FIG. 2. Contours of  $X$  and  $P$  for a single string ending on the extremal dilaton black hole. The parameters  $M = 1, Q = 1.41M, \beta = N = 1$ .